V

ESTIMATION OF THE SEQUENCE ASSOCIATED WITH THE SEQUENCE OF EXPONENTS

Both Theorems XV and XVI can be used with more precision if an estimate of the growth of the sequence $\{\Lambda_n\}$ can be given in relation to the growth of the sequence $\{\lambda_n\}$.

We shall thus prove the following two theorems.

THEOREM XVII. *If the upper density of the sequence* $\{\lambda_n\}$ *is* D, *and if*

$$\liminf_{n=\infty} (\lambda_{n+1} - \lambda_n) = h > 0, \tag{79}$$

then to every $\epsilon > 0$ *there corresponds a quantity* $A(\epsilon)$ *such that, on denoting by* $\{\Lambda_n\}$ *the sequence associated with* $\{\lambda_n\}$, *we have*

$$\Lambda_n \leqq A(\epsilon) e^{(B(D,h)+\epsilon)\lambda_n}, \quad (n \geqq 1), \tag{80}$$

where

$$B(D, h) = 3D(3 - \log(hD)) \; (\textit{with } B(0, h) = \lim_{D=0} B(D, h) = 0).$$

REMARK. It is obvious from the definition of D and h that $Dh \leqq 1$.

THEOREM XVIII. *If the upper density of the sequence* $\{\lambda_n\}$ *is finite, and if there exists a finite positive quantity* μ *such that for* n *sufficiently large:*

$$\lambda_{n+1} - \lambda_n > \lambda_n^{-\mu}, \tag{81}$$

then to every $\epsilon > 0$ *there corresponds a quantity* $A(\epsilon)$ *such that*

$$\Lambda_n \leqq A(\epsilon) e^{\lambda_n^{1+\epsilon}}, \quad (n \geqq 1). \tag{82}$$

Theorem XVII is an immediate consequence of a theorem of A. Ostrowski [**16**], (here Lemma VII) for which we give the proof given by V. Bernstein [**3**]. Theorem XVIII is a simple consequence of a classical theorem of Borel [**19**] on canonical products (here Lemma VIII).

LEMMA VII. *If the relationship* (79) *holds, the entire function* $\Lambda(z)$ *given by* (42) *has the following property: to* $\epsilon' > 0$ *and* $q > 0$ *arbitrary, there corresponds a quantity* $R(\epsilon', q)$ *such that, on denoting by* D *the upper density of* $\{\lambda_n\}$,

$$|\Lambda(z)| > e^{-(B(D,h)+\epsilon')|z|},$$

for each z *satisfying the set of inequalities* $|z| > R(\epsilon', q)$, $|z \mp \lambda_n| > q$, $(n \geqq 1)$.

From (79) and from the relationship $Dh \leqq 1$ it follows that $D \leqq \frac{1}{h} < \infty$.

Let ω be an arbitrary quantity such that $0 < \omega < h$. For $n > n_1 = n_1(\omega)$ we have $\lambda_{n+1} - \lambda_n > h - \omega$. For $x > 0$, let us set $\xi = \frac{x}{h-\omega}$. Let us also set $\mu_n = \frac{\lambda_n}{h-\omega}$, and suppose that $0 < q < \frac{h-\omega}{2}$. If $|z \pm \lambda_n| \geqq q$, and if $\lambda_{m-1} \leqq x \leqq \lambda_m (z = x + iy)$, then, for $x > n_1 + 1$ with $n_1 + 2 < m$:

$$\frac{1}{|\Lambda(z)|} = \prod_1^\infty \left|\frac{\lambda_n^2}{\lambda_n^2 - z^2}\right|$$

$$(83) \qquad \leqq \frac{2|z|^2}{q^2} \left|\prod_1^{n_1}\left(\frac{\lambda_n^2}{\lambda_n^2 - z^2}\right)\right| \cdot \prod_{n_1+1}^{m-2} \frac{\lambda_n^2}{x^2 - \lambda_n^2} \prod_{m+1}^{\infty} \frac{\lambda_n^2}{\lambda_n^2 - x^2}$$

$$\leqq A(\omega)|z|^2 \prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2 - \mu_n^2} \prod_{m+1}^{\infty}\left(1 + \frac{\xi^2}{(\mu_n - \xi)(\mu_n + \xi)}\right),$$

since the product extended from 1 to n_1 is bounded when $x > \lambda_{m-1}$, with $m > n_1 + 2$, the quantity which bounds it depending on n_1, that is to say, on ω. If $n > n_1$ we have $\mu_{n+1} - \mu_n > 1$; let us set $\Delta_n = \underset{m \geqq n}{\text{l. u. b.}} \frac{m - n_1}{\mu_m}$. It is readily seen that Δ_n tends decreasingly to $\limsup_{m=\infty} \frac{m-n_1}{\mu_m} = \Delta = (h-\omega)D$.

We have therefore:

$$\prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2 - \mu_n^2} \leqq \prod_{n_1+1}^{m-2} \frac{\mu_n}{\xi - \mu_n} \leqq \frac{\mu_{m-2}}{(m - n_1 - 2)!} \leqq \left(\frac{\mu_{m-2} e}{m - n_1 - 2}\right)^{m - n_1 - 2}$$

$$(84) \qquad = \left(\frac{\mu_{m-2}e}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \leqq \left(\frac{\mu_{m-2}e}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}\xi}.$$

We have $\frac{\mu_{m-2}}{m-n_1-2} \geqq \frac{1}{\Delta_{m-2}}$. Since the maximum of $x^{\frac{1}{x}}$ $(x>0)$ is at $x=e$, we see that, if $\Delta < \frac{1}{e}$, then, for m sufficiently large $\left(\text{in order that } \Delta_{m-2} < \frac{1}{e}\right)$, $\left(\frac{\mu_{m-2}}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \leqq \left(\frac{1}{\Delta_{m-2}}\right)^{\Delta_{m-2}}$, and, if $\Delta_{m-2} \geqq \Delta \geqq \frac{1}{e}$, then $\left(\frac{\mu_{m-2}}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \leqq e^{\frac{1}{e}}$.

In the first case, $\left(\Delta_{m-2} < \frac{1}{e}\right)$, we get from (84):

$$\prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2-\mu_n^2} \leqq \left(\frac{e}{\Delta_{m-2}}\right)^{\Delta_{m-2}\xi} = e^{\Delta_{m-2}\log\left(\frac{e}{\Delta_{m-2}}\right)^{\xi}} = e^{\Delta_{m-2}(1-\log\Delta_{m-2})\xi}.$$

In the second case, $\left(\Delta_{m-2} \geqq \frac{1}{e}\right)$:

$$\prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2-\mu_n^2} \leqq e^{\left(\frac{1}{e}+\Delta_{m-2}\right)\xi}.$$

Hence, since $\Delta = (h-\omega)D < 1$, and $\Delta_{m-2} < 1$ for m sufficiently large, we may write (for m large):

$$e^{\left(\frac{1}{e}+\Delta_{m-2}\right)\xi} \leqq e^{\Delta_{m-2}(2-\log\Delta_{m-2})\xi}.$$

Therefore, in each case, for m sufficiently large:

$$(85) \qquad \prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2-\mu_n^2} \leqq e^{\Delta_{m-2}(2-\log\Delta_{m-2})\xi}.$$

Let us now estimate the product in (83) extended from $m+1$ to ∞. Let μ_k be the largest μ_n smaller than 2ξ, if $2\xi > \mu_{m+1}$, and let $k=m$ if $2\xi \leqq \mu_{m+1}$. Let us set $E(\xi)+1=\zeta$, $E(2\Delta_k\xi)=\alpha\zeta$. We have $\alpha\zeta \geqq 2\Delta_k\xi \geqq 2\xi\frac{k-n_1}{\mu_k} \geqq k-n_1$.

We have[1]:

[1]If $k=m$ the product $\prod\limits_{m+1}^{k}$ should be replaced by unity.

$$Q(\xi)=\prod_{m+1}^{\infty}\left(1+\frac{\xi^2}{(\mu_n-\xi)(\mu_n+\xi)}\right)\leqq\prod_{m+1}^{\infty}\left(1+\frac{4}{3}\xi^2\left(\frac{\Delta_k}{n-k}\right)^2\right)$$

$$=\prod_{\nu=1}^{\alpha\zeta}\left(1+\frac{\xi}{\nu}\right)\prod_{\nu=1}^{\infty}\left(1+\frac{4}{3}\xi^2\left(\frac{\Delta_k}{\nu}\right)^2\right)\leqq\frac{((1+\alpha)\,\xi)\,!}{(\alpha\zeta)\,!\,\xi\,!}\frac{\sin(\frac{2}{\sqrt{3}}\pi i\Delta_k\xi)}{\frac{2}{\sqrt{3}}\pi i\Delta_k\xi}$$

$$\leqq B\frac{((1+\alpha)\,\xi)^{(1+\alpha)\zeta}}{(\alpha\zeta)^{\alpha\zeta}\zeta^{\zeta}}e^{\frac{2}{\sqrt{3}}\pi\Delta_k\xi}=B\left(\frac{1+\alpha}{\alpha}\right)^{\alpha\zeta}(1+\alpha)^{\zeta}e^{\frac{2\pi}{\sqrt{3}}\Delta_k\xi}$$

(B const.).

By definition, $\alpha<2\Delta_k$. If therefore we suppose $\Delta_k<\frac{1}{2}$, we shall have:

$$Q(\xi)\leqq Ce^{(2\alpha-\alpha\log\alpha+\frac{2}{\sqrt{3}}\pi\Delta_k)\xi}\quad (\text{C const.}).$$

The expression $2\alpha-\alpha\log\alpha$ increases as α increases from 0 to e, thus, if $\alpha<2\Delta_k\leqq1$:

$$Q(\xi)\leqq Ce^{(4\Delta_k-2\Delta_k\log(2\Delta_k)+\frac{2\pi}{\sqrt{3}}\Delta_k)\xi}\leqq Ce^{(7\Delta_k-2\Delta_k\log\Delta_k)\xi}. \tag{86}$$

We have, on the other hand, for ξ sufficiently large:

$$Q(\xi)\leqq\prod_{n=m+1}^{\infty}\left(1+\frac{\xi^2}{\mu_n(n-m)}\right)\leqq\prod_{\nu=1}^{\infty}\left(1+\left(\frac{\xi\sqrt{\Delta_m}}{\nu}\right)^2\right)$$

$$=\frac{\sin(\pi i\xi\sqrt{\Delta_m})}{\Delta i\xi\sqrt{\Delta_m}}\leqq e^{\xi\sqrt{\Delta+2\omega}}.$$

Thus, if ω is sufficiently small, that is to say if n_1, and thus if m, is large, that is to say if ξ is sufficiently large, $Q(\xi)$ satisfies (86) also if $2\Delta_k>1$. In all cases we have therefore, by (83), (85), and (86), for x sufficiently large ($x>x_\omega$), with ω fixed arbitrarily small ($|z-\lambda_n|\geqq q$):

$$\frac{1}{|\Lambda(z)|}\leqq e^{(\Delta(2-\log\Delta)+\Delta(7-2\log\Delta)+\omega)\xi}$$

$$=e^{(9\Delta-3\Delta\log\Delta+\omega)\frac{x}{h-\omega}}=e^{3D(3-\log[D(h-\omega)])x+\frac{\omega}{h-\omega}x}$$

$$\leqq e^{3D(3-\log[D(h-\omega)])|z|+\frac{\omega}{h-\omega}|z|}.$$

This proves that the desired inequality holds (since $\Lambda(z)$ is even) if z is in the described region with the supplementary condition $|x| \geqq a(\epsilon')$. But if $|x| \leqq c$, $\lambda_m < c \leqq \lambda_{m+1}$, with z satisfying the conditions of the lemma, then for $n \geqq m+2$:

$$\frac{\lambda_n^2}{|\lambda_n^2 - z^2|} \leqq \frac{\lambda_n^2}{\lambda_n^2 - x^2} \leqq \frac{\lambda_n^2}{\lambda_n^2 - c^2}.$$

This proves that in the region described in the lemma, with the supplementary condition $|x| \leqq c$:

$$\frac{1}{|\Lambda(z)|} \leqq K \left| \prod_{n=1}^{m+1} \frac{\lambda_n}{\lambda_n^2 - z^2} \right|,$$

where $K = K(c)$ is a constant. This achieves obviously the proof of Lemma VII.

The proof of Theorem XVII is now immediate. On choosing $q < h$, we see, by Lemma VII, that for n sufficiently large, in the closed circle $C(\lambda_n, q)$ (since this circle contains no λ_k with $k \neq n$) the following inequality holds

$$\left| \frac{z - \lambda_n}{\Lambda(z)} \right| < q e^{[B(D,\, h) + \epsilon'](\lambda_n + q)},$$

and $$\lambda_n \Lambda_n = \frac{1}{|\Lambda'(\lambda_n)|} = \lim_{z = \lambda_n} \left| \frac{z - \lambda_n}{\Lambda(z)} \right| \leqq q e^{[B(D,\, h) + \epsilon'](\lambda_n + q)},$$

which proves Theorem XVII.

LEMMA VIII. *Let $P(z)$ be a canonical product with zeros at the points $\{p_n\}$, the exponent of convergence of $\{|p_n|\}$ being ρ_1.*[1] *To each couple of two constants $h > \rho_1$, $\epsilon' > 0$, there corresponds a quantity $R(h, \epsilon')$ such that, if $|z - p_n| > p_n^{-h}$, $(n \geqq 1)$, $|z| > R(h, \epsilon')$, the following inequality holds*:

$$|P(z)| > e^{-r^{\rho_1 + \epsilon'}}.$$

This is a classical theorem of Borel, for the proof of which we refer the reader to Valiron's book on entire functions [**19**]. One of the reasons we do not give its proof here is the fact that it serves here only for the proof of Theorem XVIII which, though interesting in itself, will not be used during these lectures.

[1]That is to say, if $\epsilon > 0$: $\sum \frac{1}{|p_n|^{\rho_1 + \epsilon}} < \infty$, $\sum \frac{1}{|p_n|^{\rho_1 - \epsilon}} = \infty$.

Let us now proceed with the proof of Theorem XVIII. On setting $p_n = \lambda_n^2$, we see, since the upper density of $\{\lambda_n\}$ is finite, that there exists a constant $\lambda > 0$, such that $p_n > \lambda n^2$, and the exponent of convergence of $\{p_n\}$ is thus not larger than $\frac{1}{2}$. By Lemma VIII, in the region outside the circles $C(p_n, p_n^{-h})$, with $h > \dfrac{1}{2}$, we have

$$\left|\prod_{n=1}^{\infty}\left(1-\frac{\zeta}{p_n}\right)\right| > e^{-|\zeta|^{\frac{1}{2}+\epsilon'}}, \tag{87}$$

if $|\zeta| > r_{\epsilon'}$.

Let us set $\nu = \max(2, \mu)$. Since for $n > n_0$, $\lambda_{n+1} - \lambda_n > \lambda_n^{-\nu}$, we have for $n > n_0 + 1$: $p_{n+1} - p_n > (\lambda_{n+1} + \lambda_n)\lambda_n^{-\nu} > 2p_n^{\frac{1-\nu}{2}}$ and $p_n - p_{n-1} > 2p_{n-1}^{\frac{1-\nu}{2}} > 2p_n^{\frac{1-\nu}{2}}$. In other words, the closed circles $C(p_n, p_n^{\frac{1-\nu}{2}})$ contain no p_k with $k \neq n$. By (87) we shall then have on the circle $C(p_n, p_n^{\frac{1-\nu}{2}})$, for n large:

$$\left|\prod_{k=1}^{\infty}\left(1-\frac{\zeta}{p_k}\right)\right| > e^{-\left(p_n + p_n^{\frac{1-\nu}{2}}\right)^{\frac{1}{2}+\epsilon'}} > e^{-(2p_n)^{\frac{1}{2}+\epsilon'}} > e^{-\lambda_n^{1+3\epsilon'}}$$

On the circumference of this circle we have then

$$\left|\frac{\zeta - p_n}{\prod_{k=1}^{\infty}\left(1-\frac{\zeta}{p_k}\right)}\right| \leqq p_n^{\frac{1-\nu}{2}} e^{\lambda_n^{1+3\epsilon'}}$$

We have thus for $n > n(\epsilon')$:

$$\lim_{\zeta = p_n}\left|\frac{\zeta - p_n}{\prod\left(1-\frac{\zeta}{p_k}\right)}\right| = \lim_{z=\lambda_n}\left|\frac{z^2-\lambda_n^2}{\Lambda(z)}\right| = 2\lambda_n^2\Lambda_n \leqq \lambda_n^{1-\nu} e^{\lambda_n^{1+3\epsilon'}},$$

which proves Theorem XVIII.

In the inequalities (54) or (78) the quantities Λ_n can be replaced by the right-hand expression of the inequality (80), if the λ_n satisfy the relationship (79) (in addition, of course, to the relationships required by Theorems XV or XVI), and by the right-hand expression of the inequality (82), if the λ_n satisfy the supplementary condition (81).